

# VISIBLE LATTICE POINTS AND THE CHROMATIC ZETA FUNCTION OF A GRAPH

JAVIER CILLERUELO

**ABSTRACT.** We study the probability that a random polygon of  $k$  vertices in the lattice  $\{1, \dots, n\}^s$  does not contain more lattice points than the  $k$  vertices of the polygon. Then we introduce the chromatic zeta function of a graph to generalize this problem to other configurations induced by a given graph  $\mathcal{H}$ .

## 1. INTRODUCTION

Two distinct points  $X, Y$  of the  $s$ -dimensional integer lattice are said to be mutually visible if the line segment joining them contains no other lattice point. We denote this situation by  $X \diamond Y$ . It is well known [1] that if  $X, Y$  are lattice points taken at random uniformly in  $[1, n]^s$  then  $\mathbb{P}(X \diamond Y) \sim \zeta^{-1}(s)$  as  $n \rightarrow \infty$ , where  $\zeta(s)$  is the classical Riemann zeta function. Since  $X \diamond Y$  and  $Y \diamond Z$  are independent events, then  $\mathbb{P}(X \diamond Y \diamond Z) \sim \zeta^{-2}(s)$ . What about  $\mathbb{P}(X \diamond Y \diamond Z \diamond X)$ ? In other words, what is the probability that the three edges of a random triangle  $X, Y, Z$  contains no other lattice points than their vertices?

At first sight, we could expect that  $\mathbb{P}(X \diamond Y \diamond Z \diamond X) \sim \zeta^{-3}(s)$  since apparently the three events,  $X \diamond Y$ ,  $Y \diamond Z$ ,  $Z \diamond X$ , are independent events. We prove that this intuition is not correct. In fact we obtain a more general result.

**Theorem 1.1.** *Let  $s, k \geq 2$  positive integers. If  $X^1, \dots, X^k$  are lattice points taken uniformly at random in  $[1, n]^s$ , we have*

$$\lim_{n \rightarrow \infty} \mathbb{P}(X^1 \diamond X^2 \diamond \dots \diamond X^k \diamond X^1) = \zeta^{-k}(s) \prod_p \left( 1 + \frac{(-1)^k}{(p^s - 1)^{k-1}} \right).$$

The repetition of vertices  $X^i$  is allowed in Theorem 1.1. However, the probability of these degenerate cases tends to zero as  $n \rightarrow \infty$ , so we could have formulated Theorem 1.1 saying that  $X^1, \dots, X^k$  are distinct lattice points.

It is interesting to note that the value of the limit in Theorem 1.1 is smaller than  $\zeta^{-k}(s)$  when  $k$  is odd and greater than  $\zeta^{-k}(s)$  when  $k$  is even. We do not understand the reason of this phenomenon.

The following version of Theorem 1.1 can be more illustrative. Take a lattice point  $X^1$  at random. Then take a random lattice point  $X^2$  visible from  $X^1$ , then take a random lattice point  $X^3$  visible from  $X^2$  and so on. What is the probability that  $X^k$  is visible from  $X^1$ ? Corollary 1.1, which is a trivial consequence of Theorem 1.1, answers this question.

---

1991 *Mathematics Subject Classification.* 2000 Mathematics Subject Classification: 05C31.

*Key words and phrases.* visible lattice points, chromatic polynomial, zeta function.

This work has been supported by MINECO project MTM2014-56350-P and ICMAT Severo Ochoa project SEV-2011-0087.

**Corollary 1.1.** *Let  $s, k \geq 2$  positive integers. If  $X^1, \dots, X^k$  are lattice points taken uniformly at random in  $[1, n]^s$  we have*

$$\lim_{n \rightarrow \infty} \mathbb{P}(X^k \diamond X^1 / X^1 \diamond X^2 \diamond \dots \diamond X^k) = \zeta^{-1}(s) \prod_p \left( 1 + \frac{(-1)^k}{(p^s - 1)^{k-1}} \right).$$

Again, we see that  $\mathbb{P}(X^k \diamond X^1 / X^1 \diamond X^2 \diamond \dots \diamond X^k)$  is smaller than  $\mathbb{P}(X^k \diamond X^1)$  when  $k$  is odd and greater when  $k$  is even. Theorem 1.1 can be extended to more general configurations.

**Definition 1.** *Given a graph  $\mathcal{H}$  of order  $k$  we say that a sequence of lattice points  $(X^1, \dots, X^k)$  is  $\mathcal{H}$ -visible if  $X^i \diamond X^j$  whenever  $\{i, j\} \in E(\mathcal{H})$ .*

Our main Theorem is the following.

**Theorem 1.2.** *Let  $s, k \geq 2$  positive integers and  $\mathcal{H}$  a graph of order  $k$ . If  $X^1, \dots, X^k$  are lattice points taken uniformly at random in  $[1, n]^s$  then we have*

$$\lim_{n \rightarrow \infty} \mathbb{P}((X^1, \dots, X^k) \text{ is } \mathcal{H}\text{-visible}) = \zeta_{\mathcal{H}}^{-1}(s),$$

where  $\zeta_{\mathcal{H}}(s)$  is the chromatic zeta function of  $\mathcal{H}$  defined by

$$\zeta_{\mathcal{H}}(s) = \prod_p \left( \frac{P_{\mathcal{H}}(p^s)}{p^{ks}} \right)^{-1}$$

where  $P_{\mathcal{H}}$  is the chromatic polynomial of  $\mathcal{H}$ .

If we consider the linear graph  $\mathcal{H} = L_k$ , with chromatic polynomial  $P_{L_k}(x) = x(x-1)^{k-1}$ , we have that  $\zeta_{\mathcal{H}}(s) = \zeta^{k-1}(s)$  and we recover the classic result [1]:

$$\lim_{n \rightarrow \infty} \mathbb{P}(X^1 \diamond \dots \diamond X^k : X^i \in [1, n]^s) = \zeta^{-(k-1)}(s).$$

Theorem 1.1 follows from Theorem 1.2 by taking the cycle of  $k$  vertices,  $\mathcal{H} = C_k$ , and observing that  $P_{C_k}(x) = (x-1)^k + (-1)^k(x-1)$ :

$$\zeta_{C_k}^{-1}(s) = \prod_p \left( \frac{(p^s - 1)^k + (-1)^k(p^s - 1)}{p^{ks}} \right) = \prod_p \left( 1 - \frac{1}{p^s} \right)^k \prod_p \left( 1 + \frac{(-1)^k}{(p^s - 1)^{k-1}} \right).$$

David Rearick [2] considered a related problem. Given a set  $S_m = \{X^1, \dots, X^m\}$  of  $m$  mutually visible lattice points, he studied the probability that a random lattice point in  $[1, n]^s$  is visible from all the lattice points of  $S_m$ . He proved that

$$(1.1) \quad \lim_{n \rightarrow \infty} \mathbb{P}(X \in [1, n]^2 : X \diamond X^i, i = 1, \dots, m) = \prod_p \left( 1 - \frac{m}{p^s} \right)$$

if  $m < 2^s$  and 0 if  $m \geq 2^s$ . In particular (1.1) implies that if  $m < 2^s$  and  $X^1, \dots, X^{m+1}$  are taken uniformly at random in  $[1, n]^s$ , then

$$\lim_{n \rightarrow \infty} \mathbb{P}(X^1, \dots, X^{m+1} \text{ is } K_{m+1}\text{-visible} / X^1, \dots, X^m \text{ is } K_m\text{-visible}) = \prod_p \left( 1 - \frac{m}{p^s} \right).$$

This result can be obtained easily from Theorem 1.2 considering the chromatic polynomials of the complete graphs,  $P_{K_{m+1}}(x) = x(x-1) \dots (x-m)$ ,  $P_{K_m}(x) = x(x-1) \dots (x-m+1)$ , and observing that

$$\frac{\zeta_{K_{m+1}}^{-1}(s)}{\zeta_{K_m}^{-1}(s)} = \prod_p \frac{P_{K_{m+1}}(p^s)}{p^{(m+1)s}} \prod_p \frac{p^{ms}}{P_{K_m}(p^s)} = \prod_p \frac{p^s - m}{p^s} = \prod_p \left( 1 - \frac{m}{p^s} \right).$$

## 2. PROOF OF THEOREM 1.2

Given two lattice points  $X^i = (x_1^i, \dots, x_s^i)$  and  $X^j = (x_1^j, \dots, x_s^j)$  we write  $X^i \equiv X^j \pmod{p}$  if  $x_r^i \equiv x_r^j \pmod{p}$  for all  $r = 1, \dots, s$ . We write  $X^i \not\equiv X^j \pmod{p}$  otherwise.

Given a prime  $p$ , we say that  $(X^1, \dots, X^k)$  is  $\mathcal{H}_p$ -visible if  $X^i \not\equiv X^j \pmod{p}$  whenever  $\{i, j\} \in E(\mathcal{H})$ . The first observation is that

$$(2.1) \quad (X^1, \dots, X^k) \text{ is } \mathcal{H}\text{-visible} \iff (X^1, \dots, X^k) \text{ is } \mathcal{H}_p\text{-visible for any prime } p.$$

For any positive integer  $M$  and  $n > M$  we have

$$(2.2) \quad |\{X^1, \dots, X^k \in [1, n]^s : (X^1, \dots, X^k) \text{ is } \mathcal{H}_p\text{-visible for any } p\}| \\ = |\{X^1, \dots, X^k \in [1, n]^s : (X^1, \dots, X^k) \text{ is } \mathcal{H}_p\text{-visible for any } p \leq M\}| + O(|R|),$$

where

$$R = \{X^1, \dots, X^k \in [1, n]^s : (X^1, \dots, X^k) \text{ is not } \mathcal{H}_p\text{-visible for some } p > M\}.$$

We split  $R$  in two sets:  $R = R_1 \cup R_2$ . The set  $R_1$  contains those  $(X^1, \dots, X^k)$  with  $X^i = X^j$  for some  $i \neq j$  and  $R_2$  contains those with all  $X^i$  distinct.

Clearly,

$$(2.3) \quad |R_1| \leq \binom{k}{2} n^{s(k-1)}.$$

On the other hand we observe that if  $X^i \neq X^j$  then  $X^i \not\equiv X^j \pmod{p}$  for  $p \geq n$ , so  $(X^1, \dots, X^k)$  is always  $\mathcal{H}_p$ -visible when  $p \geq n$  for those  $(X^1, \dots, X^k)$  counted in  $R_2$ . Indeed, for a fixed  $X^i = (x_1^i, \dots, x_s^i)$  the number of  $X^j = (x_1^j, \dots, x_s^j) \in [1, n]^s$  such that  $X^j \equiv X^i \pmod{p}$  is  $(n/p + O(1))^s \ll n^s/p^s$  for  $p < n$ . Thus,

$$\begin{aligned} |R_2| &\leq \sum_{M < p < n} |\{\text{distinct } X^1, \dots, X^k \in [1, n]^s : (X^1, \dots, X^k) \text{ is not } \mathcal{H}_p\text{-visible}\}| \\ &\leq \sum_{M < p < n} |\{\text{distinct } X^1, \dots, X^k \in [1, n]^s : X^i \equiv X^j \pmod{p} \text{ for some } i \neq j\}| \\ &\leq \sum_{M < p < n} \binom{k}{2} |\{\text{distinct } X^1, \dots, X^k \in [1, n]^s : X^1 \equiv X^2 \pmod{p}\}| \\ &\ll \sum_{M < p < n} \frac{n^{ks}}{p^s} \end{aligned}$$

and we get the upper bound

$$(2.4) \quad |R_2| \ll n^{ks} M^{1-s}.$$

By (2.2), (2.3) and (2.4) we have

$$(2.5) \quad |\{X^1, \dots, X^k \in [1, n]^s : (X^1, \dots, X^k) \text{ is } \mathcal{H}\text{-visible}\}| \\ = |\{X^1, \dots, X^k \in [1, n]^s : (X^1, \dots, X^k) \text{ is } \mathcal{H}_p\text{-visible for any } p \leq M\}| \\ + O(n^{s(k-1)}) + O(n^{ks} M^{1-s}).$$

The next step is to estimate the quantity

$$(2.6) \quad |\{X^1, \dots, X^k \in [1, n]^s : (X^1, \dots, X^k) \text{ is } \mathcal{H}_p\text{-visible for any } p \leq M\}|.$$

A good coloration of a labeled graph  $\mathcal{H}$  is an assignment of colours to the vertices such that two adjacent vertices do not share the same colour. The polynomial chromatic  $P_{\mathcal{H}}(x)$  counts the number of good colorations of  $\mathcal{H}$  using  $x$  colours.

For each  $p$  we assign to each vertex  $X = (x_1, \dots, x_s)$  the  $p$ -colour  $c_p(X)$  defined as the only vector  $c_p(X) \in [0, p-1]^s$  such that  $c_p(X) \equiv X \pmod{p}$ .

We observe that  $(X^1, \dots, X^k)$  is  $\mathcal{H}_p$ -visible if and only if there exists a good  $p$ -coloration  $C_p = (c_p^1, \dots, c_p^k)$  of  $\mathcal{H}$  such that  $(c_p(X^1), \dots, c_p(X^k)) = C_p$ .

Thus,  $(X^1, \dots, X^k)$  is  $\mathcal{H}_p$ -visible for any  $p \leq M$  if and only if there exists a sequence of good colorations  $(C_p)_{p \leq M}$  such that  $(c_p(X^1), \dots, c_p(X^k)) = C_p$  for all  $p \leq M$ .

Since for each prime  $p$  there are  $p^s$  colours, the number of good  $p$ -colorations of  $\mathcal{H}$  is  $P_{\mathcal{H}}(p^s)$ , where  $P_{\mathcal{H}}$  is the chromatic polynomial of  $\mathcal{H}$ . Therefore, the number of sequences of good colorations  $(C_p)_{p \leq M}$  is

$$(2.7) \quad \prod_{p \leq M} P_{\mathcal{H}}(p^s).$$

Thus we have

$$(2.8) \quad |\{X^1, \dots, X^k \in [1, n]^s : (X^1, \dots, X^k) \text{ is } \mathcal{H}_p\text{-visible for any } p \leq M\}| \\ = \sum^* |\{X^1, \dots, X^k \in [1, n]^s : (c_p(X^1), \dots, c_p(X^k)) = C_p, p \leq M\}|$$

where the sum  $\sum^*$  is extended over all sequences of good colorations  $(C_p)_{p \leq M}$  of the graph  $\mathcal{H}$ .

Given a sequence of colorations  $(C_p)_{p \leq M} = (c_p^1, \dots, c_p^k)_{p \leq M}$  we have that

$$(2.9) \quad |\{X^1, \dots, X^k \in [1, n]^s : c_p(X^i) = c_p^i, i = 1, \dots, k, \text{ for all } p \leq M\}| \\ = \prod_{i=1}^k |\{X \in [1, n]^s : c_p(X) = c_p^i, \text{ for all } p \leq M\}|.$$

Given the vectors  $c_p^i = (c_{p1}^i, \dots, c_{ps}^i)$ ,  $p \leq M$ , the lattice points  $X = (x_1, \dots, x_s)$  with  $c_p(X) = c_p^i$  for all  $p \leq M$  will be those such that the congruences  $x_r \equiv c_{pr}^i \pmod{p}$ ,  $p \leq M$  hold for any  $r = 1, \dots, s$ . By the Chinese Remainder Theorem these congruences are equivalent, for each  $r = 1, \dots, s$ , to the congruence  $x_r \equiv a_r \pmod{\prod_{p \leq M} p}$  for some  $a_r$ . The number of  $x_r \leq n$  satisfying each congruence is  $\frac{n}{\prod_{p \leq M} p} + O(1)$ , so the number of  $X \in [1, n]^s$  with  $c_p(X) = c_p^i$  for all  $p \leq M$  is

$$\left( \frac{n}{\prod_{p \leq M} p} + O(1) \right)^s.$$

Since this estimate does not depend on the values of  $c_p^i$  we have

$$(2.10) \quad \prod_{i=1}^k |\{X \in [1, n]^s : c_p(X) = c_p^i \text{ for all } p \leq M\}| = \left( \frac{n}{\prod_{p \leq M} p} + O(1) \right)^{sk}.$$

Summing up, as consequence of (2.8), (2.9), (2.10) and (2.7) we obtain

$$|\{X^1, \dots, X^k \in [1, n]^s : (X^1, \dots, X^k) \text{ is } \mathcal{H}_p\text{-visible for any } p \leq M\}| \\ = \left( \frac{n}{\prod_{p \leq M} p} + O(1) \right)^{sk} \times |\{\text{sequences of good colorations } (c_p^1, \dots, c_p^k), p \leq M\}| \\ = \left( \frac{n}{\prod_{p \leq M} p} + O(1) \right)^{sk} \prod_{p \leq M} P_{\mathcal{H}}(p^s) = n^{sk} \left( \prod_{p \leq M} \frac{P_{\mathcal{H}}(p^s)}{p^{sk}} \right) \left( 1 + O\left( \frac{\prod_{p \leq M} p}{n} \right) \right)^{sk}.$$

In terms of probability we have proved that

$$\begin{aligned} & \mathbb{P}(\{X^1, \dots, X^k \in [1, n]^s : (X^1, \dots, X^k) \text{ is } \mathcal{H}_p\text{-visible for any } p \leq M\}) \\ &= \left( \prod_{p \leq M} \frac{P_{\mathcal{H}}(p^s)}{p^{sk}} \right) \left( 1 + O\left(\frac{\prod_{p \leq M} p}{n}\right) \right)^{sk}. \end{aligned}$$

Using (2.5) we have that

$$\begin{aligned} & \mathbb{P}(\{X^1, \dots, X^k \in [1, n]^s : (X^1, \dots, X^k) \text{ is } \mathcal{H}\text{-visible}\}) \\ &= \left( \prod_{p \leq M} \frac{P_{\mathcal{H}}(p^s)}{p^{sk}} \right) \left( 1 + O\left(\frac{\prod_{p \leq M} p}{n}\right) \right)^{sk} + O(n^{-s}) + O(M^{1-s}). \end{aligned}$$

Taking the limit as  $n \rightarrow \infty$  we get

$$\lim_{n \rightarrow \infty} \mathbb{P}(\{X^1, \dots, X^k \in [1, n]^s : (X^1, \dots, X^k) \text{ is } \mathcal{H}\text{-visible}\}) = \prod_{p \leq M} \frac{P_{\mathcal{H}}(p^s)}{p^{sk}} + O(M^{1-s}).$$

Finally, taking the limit as  $M \rightarrow \infty$  we have

$$\lim_{n \rightarrow \infty} \mathbb{P}(\{X^1, \dots, X^k \in [1, n]^s : (X^1, \dots, X^k) \text{ is } \mathcal{H}\text{-visible}\}) = \prod_p \frac{P_{\mathcal{H}}(p^s)}{p^{sk}} = \zeta_{\mathcal{H}}^{-1}(s).$$

#### REFERENCES

- [1] J. Christopher, The asymptotic of some  $k$ -dimensional sets. *Amer. Math. Monthly* 63 (1956), 399–401.
- [2] D. F. Rearick, Mutually visible lattice points. *Norske Vid. Selsk. Forh. (Trondheim)* 39 (1966) 41–45.

INSTITUTO DE CIENCIAS MATEMÁTICAS (ICMAT) AND DEPARTAMENTO DE MATEMÁTICAS, UNIVERSIDAD AUTÓNOMA DE MADRID, 28049 MADRID, SPAIN